Towards the Model and Fixpoint Semantics

- We have seen previously the operational semantics (SLD-resolution).
- We now present the (declarative) Model Semantics:
  - We define our semantic domain (Herbrand interpretations).
  - We introduce the Minimal Herbrand Model.
- And the (also declarative) Fixpoint Semantics.
  - We recall some basic fixpoint theory.
  - Present the $T_P$ operator and the classic fixpoint semantics.
Declarative Semantics – Herbrand Base and Universe

- Given a first-order language $L$, with a non-empty set of variables, constants, function symbols, relation symbols, connectives, quantifiers, etc. and given a syntactic object $A$, 

  \[ \text{ground}(A) = \{ A\theta | \exists \theta \in \text{Subst}, \text{var}(A\theta) = \emptyset \} \]

  i.e. the set of all "ground instances" of $A$.

- Given $L$, $U_L$ (Herbrand universe) is the set of all ground terms of $L$.

- $B_L$ (Herbrand Base) is the set of all ground atoms of $L$.

- Similarly, for the language $L_P$ associated with a given program $P$ we define $U_P$, and $B_P$.

Declarative Semantics – Herbrand Base and Universe (example)

- Program:

  $P = \{ \begin{array}{l} p(f(X)) \leftarrow p(X). \\
  p(a). \\
  q(a). \\
  q(b). \end{array} \}$

- Herbrand universe:

  $U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \}$

- Herbrand base:

  $B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \}$
Herbrand Interpretations and Models

- A **Herbrand Interpretation** is a subset of $B_L$, i.e. the set of all Herbrand interpretations $I_L = \wp(B_L)$.
  (Note that $I_L$ forms a complete lattice under $\subseteq$ – important for fixpoint operations to be introduced later).

- In previous example: $P = \{ \ p(f(X)) \leftarrow p(X). \ p(a). \ q(a). \ q(b). \ \}$
  $U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \}$
  $B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \}$
  $I_P = \text{all subsets of } B_P$

- A **Herbrand Model** is a Herbrand interpretation which contains all logical consequences of the program.

- The **Minimal Herbrand Model** $H_P$ is the smallest Herbrand interpretation which contains all logical consequences of the program. (Theorem: it is unique.)

- Example:
  $H_P = \{ q(a), q(b), p(a), p(f(a)), p(f(f(a))), \ldots \}$

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Declarative Semantics, Completeness, Correctness

- **Declarative semantics of a logic program $P$**: the set of ground facts which are logical consequences of the program (i.e., $H_P$).
  (I.e., the Minimal Herbrand model (or “least model”) of $P$).

- **Intended meaning of a logic program $P$**: the set $I$ of ground facts that the user expects to be logical consequences of the program.

- A logic program is **correct** if $H_P \subseteq I$.

- A logic program is **complete** if $I \subseteq H_P$.

- Example:
  `father(john,peter).`
  `father(john,mary).`
  `mother(mary,mike).`
  `grandfather(X,Y) ← father(X,Z), father(Z,Y).`

  with the usual intended meaning is correct but incomplete.
Towards a Fixpoint Semantics for LP – Fixpoint Basics

- A fixpoint for an operator $T : X \rightarrow X$ is an element of $x \in X$ such that $x = T(x)$.
- If $X$ is a poset, $T$ is monotonic if $\forall x, y \in X, \, x \leq y \Rightarrow T(x) \leq T(y)$
- If $X$ is a complete lattice and $T$ is monotonic the set of fixpoints of $T$ is also a complete lattice [Tarski]
- The least element of the lattice is the least fixpoint of $T$, denoted $lfp(T)$
- Powers of a monotonic operator (successive applications):
  $T \uparrow 0(x) = x$
  $T \uparrow n(x) = T(T \uparrow (n-1)(x)) (n \text{ is a successor ordinal})$
  $T \uparrow \omega(x) = \bigcup \{T \uparrow n(x) | n < \omega\}$
  We abbreviate $T \uparrow \alpha(\bot)$ as $T \uparrow \alpha$
- There is some $\omega$ such that $T \uparrow \omega = lfp(T)$. The sequence $T \uparrow 0, T \uparrow 1, \ldots, lfp T$ is the Kleene sequence for $T$
- In a finite lattice the Kleene sequence for a monotonic operator $T$ is finite

Towards a Fixpoint Semantics for LP – Fixpoint Basics (Contd.)

- A subset $Y$ of a poset $X$ is an (ascending) chain iff $\forall y, y' \in Y, \, y \leq y' \lor y' \leq y$
- A complete lattice $X$ is ascending chain finite (or Noetherian) if all ascending chains are finite
- In an ascending chain finite lattice the Kleene sequence for a monotonic operator $T$ is finite
A Fixpoint Semantics for Logic Programs

- **Semantic domain:** $I_L = \wp(B_L)$.
- I.e., the elements of the semantic domain and interpretations (subsets of the Herbrand base).
- **Semantic operator** (defined on programs):
  the *immediate consequences operator*, $T_P$:
  
  $T_P$ is a mapping: $T_P : I_P \rightarrow I_P$ defined by:
  
  $T_P(I) = \{ A \in B_P \mid \exists C \in \text{ground}(P), C = A \leftarrow L_1, \ldots, L_n \text{ and } L_1, \ldots L_n \in I \}$
  
  (in particular, if $(A \leftarrow) \in P$, then every element of $\text{ground}(A)$ is in $T_P(I), \forall I$).
- $T_P$ is monotonic, so:
  
  - it has a least fixpoint $I^*$ so that $T_P(I^*) = I^*$,
  - this fixpoint can be obtained by applying $T_P$ iteratively starting from the bottom element of the lattice (the empty interpretation).
A Fixpoint Semantics for Logic Programs: Example 1 (finite)

\[ P = \{ p(X, a) \leftarrow q(X). \]
\[ p(X, Y) \leftarrow q(X), r(Y). \]
\[ q(a). \]
\[ r(b). \]
\[ q(b). \]
\[ r(c). \}

\[ U_P = \{ a, b, c \} \]
\[ B_P = \{ p(a, a), p(a, b), p(a, c), p(b, a), p(b, b), p(b, c), p(c, a), p(c, b), p(c, c), \]
\[ q(a), q(b), q(c), \]
\[ r(a), r(b), r(c) \}\]

\[ I_P = \text{all subsets of } B_P \]
\[ H_P = \{ q(a), q(b), r(b), r(c), p(a, a), p(b, a), p(b, b), p(a, c), p(b, c) \}\]

\[ T_P \uparrow 0 = \{ q(a), q(b), r(b), r(c) \} \]
\[ T_P \uparrow 1 = \{ q(a), q(b), r(b), r(c) \} \cup \{ p(a, a), p(b, a), p(b, b), p(a, c), p(b, c) \} \]
\[ T_P \uparrow 2 = T_P \uparrow 1 = \text{lfp}(T_P) = H_P \]

A Fixpoint Semantics for Logic Programs: Example 2 (infinite)

\[ P = \{ p(f(X)) \leftarrow p(X). \]
\[ p(a). \]
\[ q(a). \]
\[ q(b). \}\]

\[ U_P = \{ a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots \} \]
\[ B_P = \{ p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \ldots \} \]
\[ I_P = \text{all subsets of } B_P \]
\[ H_P = \{ q(a), q(b), p(a) \} \cup \{ p(f^n(a)) \mid n \in \mathbb{N} \} \]
\[ \text{where we define } f^n(a) \text{ to be } f \text{ nested } n \text{ times and then applied to } a. \]
\[ \text{(i.e., } q(a), q(b), p(a), p(f(a)), p(f(f(a))), p(f(f(f(a)))), \ldots \}\]

\[ T_P \uparrow 0 = \{ p(a), q(a), q(b) \} \]
\[ T_P \uparrow 1 = \{ p(a), q(a), q(b), p(f(a)) \} \]
\[ T_P \uparrow 2 = \{ p(a), q(a), q(b), p(f(a)), p(f(f(a))) \} \]
\[ \ldots \]
\[ T_P \uparrow \omega = H_P \]
A Fixpoint Semantics for Logic Programs: Example 3 (infinite)

- Example:
  \[ P = \{ \text{nat}(0), \]
  \[ \text{nat}(s(X)) \leftarrow \text{nat}(X). \]
  \[ \text{sum}(0, X, X), \]
  \[ \text{sum}(s(X), Y, s(Z)) \leftarrow \text{sum}(X, Y, Z). \} \]
  \[ U_P = \{0\} \cup \{s(x) \mid x \in U_P\} \]
  (i.e., \{0, s(0), s(s(0)), s(s(s(0))), \ldots\}).

\[ B_P = \{\text{nat}(x) \mid x \in U_P\} \cup \{\text{sum}(x, y, z) \mid x, y, z \in U_P\} \]
(i.e., \{\text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), \ldots\} \cup \{\text{sum}(0, 0, 0), \text{sum}(s(0), 0, 0), \text{sum}(0, s(0), 0), \text{sum}(0, 0, s(0)), \ldots\}).

A Fixpoint Semantics for Logic Programs: Example 3 (infinite, cont.)

Constructing the least fixpoint of the \( T_P \) operator:

\[ T_P \uparrow 0 = \{\text{nat}(0)\} \cup \{\text{sum}(0, x, x) \mid x \in U_P\} \]
\[ T_P \uparrow 1 = T_P \uparrow 0 \cup \{\text{nat}(s(0))\} \]
\[ \quad \cup \{\text{sum}(s(0), y, s(y)) \mid y \in U_P\} \]
\[ T_P \uparrow 2 = T_P \uparrow 1 \cup \{\text{nat}(s(s(0)))\} \]
\[ \quad \cup \{\text{sum}(s(s(0)), y, s(s(y))) \mid y \in U_P\} \]
\[ T_P \uparrow 3 = T_P \uparrow 2 \cup \{\text{nat}(s(s(s(0))))\} \]
\[ \quad \cup \{\text{sum}(s(s(s(0))), y, s(s(s(y)))) \mid y \in U_P\} \]
...

\[ T_P \uparrow \omega = \{\text{nat}(x) \mid x \in U_P\} \cup \{\text{sum}(s^n(0), y, s^n(y)) \mid y \in U_P \land n \in \mathbb{N}\} \]

where we define \( s^n(y) \) to be \( s \) nested \( x \) times and then applied to \( y \).
(Characterization Theorem) [Van Emden and Kowalski]
A program $P$ has a Herbrand model $H_P$ such that:
- $H_P$ is the least Herbrand Model of $P$.
- $H_P$ is the least fixpoint of $T_P$ ($lfp T_P$).
- $H_P = T_P \uparrow \omega$.

I.e., least model semantics ($H_P$) $\equiv$ fixpoint semantics ($lfp T_P$)

- In addition, there is also an equivalence with the operational semantics (SLD-resolution):
  - SLD-resolution answers "yes" to $a \in B_P \iff a \in H_P$.

- Because it gives us a way to directly build $H_P$ (for finite models), the least fixpoint semantics can in some cases also be an operational semantics (e.g., for datalog in deductive databases).