

Computational Logic
Logic Programming:
Model and Fixpoint Semantics

1

Towards the Model and Fixpoint Semantics

- We have seen previously the operational semantics (SLD-resolution).
- We now present the (declarative) *Model Semantics*:
 - ◇ We define our semantic *domain* (Herbrand interpretations).
 - ◇ We introduce the Minimal Herbrand Model.
- And the (also declarative) *Fixpoint Semantics*.
 - ◇ We recall some basic fixpoint theory.
 - ◇ Present the T_P operator and the classic fixpoint semantics.

2

Declarative Semantics – Herbrand Base and Universe

- Given a first-order language L , with a non-empty set of variables, constants, function symbols, relation symbols, connectives, quantifiers, etc. and given a syntactic object A ,

$$\text{ground}(A) = \{A\theta \mid \exists \theta \in \text{Subst}, \text{var}(A\theta) = \emptyset\}$$

i.e. the set of all “ground instances” of A .

- Given L , U_L (*Herbrand universe*) is the set of all ground terms of L .
- B_L (*Herbrand Base*) is the set of all ground atoms of L .
- Similarly, for the language L_P associated with a given program P we define U_P , and B_P .

Declarative Semantics – Herbrand Base and Universe (example)

- Program:

$$P = \{ p(f(X)) \leftarrow p(X). \\ p(a). \\ q(a). \\ q(b). \}$$

- Herbrand universe:

$$U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \dots\}$$

- Herbrand base:

$$B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \dots\}$$

Herbrand Interpretations and Models

- A *Herbrand Interpretation* is a subset of B_L , i.e. the set of all Herbrand interpretations $I_L = \wp(B_L)$.
(Note that I_L forms a *complete lattice* under \subseteq – important for fixpoint operations to be introduced later).
- In previous example: $P = \{ p(f(X)) \leftarrow p(X). \quad p(a). \quad q(a). \quad q(b). \quad \}$
 $U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \dots\}$
 $B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \dots\}$
 $I_P = \text{all subsets of } B_P$
- A *Herbrand Model* is a Herbrand interpretation which contains all logical consequences of the program.
- The *Minimal Herbrand Model* H_P is the smallest Herbrand interpretation which contains all logical consequences of the program. (Theorem: it is unique.)
- Example:
 $H_P = \{q(a), q(b), p(a), p(f(a)), p(f(f(a))), \dots\}$

5

Declarative Semantics, Completeness, Correctness

- *Declarative semantics of a logic program P*:
the set of ground facts which are logical consequences of the program (i.e., H_P).
(I.e., the *Minimal Herbrand model* (or “least model”) of P).
- *Intended meaning of a logic program P*:
the set I of ground facts that the user expects to be logical consequences of the program.
- A logic program is *correct* if $H_P \subseteq I$.
- A logic program is *complete* if $I \subseteq H_P$.
- Example:
father(john,peter).
father(john,mary).
mother(mary,mike).
grandfather(X,Y) \leftarrow father(X,Z), father(Z,Y).
with the usual intended meaning is *correct* but *incomplete*.

6

Towards a Fixpoint Semantics for LP – Fixpoint Basics

- A *fixpoint* for an operator $T : X \rightarrow X$ is an element of $x \in X$ such that $x = T(x)$.
- If X is a poset, T is monotonic if $\forall x, y \in X, x \leq y \Rightarrow T(x) \leq T(y)$
- If X is a complete lattice and T is monotonic the set of fixpoints of T is also a complete lattice [Tarski]
- The least element of the lattice is the *least fixpoint* of T , denoted $lfp(T)$
- Powers of a monotonic operator (successive applications):

$$T \uparrow 0(x) = x$$

$$T \uparrow n(x) = T(T \uparrow (n-1)(x)) \text{ (} n \text{ is a successor ordinal)}$$

$$T \uparrow \omega(x) = \sqcup \{T \uparrow n(x) \mid n < \omega\}$$

We abbreviate $T \uparrow \alpha(\perp)$ as $T \uparrow \alpha$

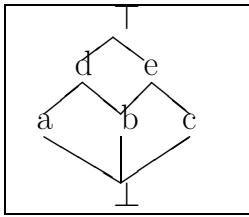
- There is some ω such that $T \uparrow \omega = lfp T$. The sequence $T \uparrow 0, T \uparrow 1, \dots, lfp T$ is the *Kleene sequence* for T
- In a finite lattice the Kleene sequence for a monotonic operator T is finite

Towards a Fixpoint Semantics for LP – Fixpoint Basics (Contd.)

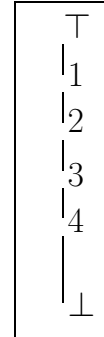
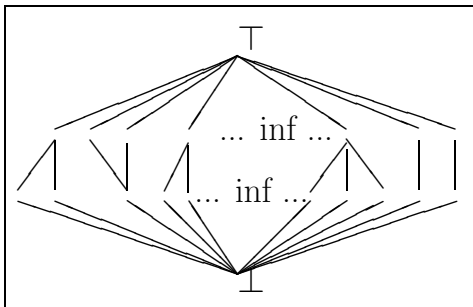
- A subset Y of a poset X is an (ascending) chain iff $\forall y, y' \in Y, y \leq y' \vee y' \leq y$
- A complete lattice X is *ascending chain finite* (or *Noetherian*) if all ascending chains are finite
- In an ascending chain finite lattice the Kleene sequence for a monotonic operator T is finite

Lattice Structures

finite



finite_depth



ascending chain finite

9

A Fixpoint Semantics for Logic Programs

- Semantic *domain*: $I_L = \wp(B_L)$.
- I.e., the elements of the semantic domain and *interpretations* (subsets of the Herbrand base).
- Semantic *operator* (defined on programs):
the *immediate consequences operator*, T_P :
 - ◇ T_P is a mapping: $T_P : I_P \rightarrow I_P$ defined by:
$$T_P(I) = \{A \in B_P \mid \exists C \in \text{ground}(P), C = A \leftarrow L_1, \dots, L_n \text{ and } L_1, \dots, L_n \in I\}$$
(in particular, if $(A \leftarrow) \in P$, then every element of $\text{ground}(A)$ is in $T_P(I)$, $\forall I$).
- T_P is monotonic, so:
 - ◇ it has a least fixpoint I^* so that $T_P(I^*) = I^*$,
 - ◇ this fixpoint can be obtained by applying T_P iteratively starting from the bottom element of the lattice (the empty interpretation).

10

A Fixpoint Semantics for Logic Programs: Example 1 (finite)

$$P = \{ p(X, a) \leftarrow q(X). \\ p(X, Y) \leftarrow q(X), r(Y). \\ q(a). \quad r(b). \\ q(b). \quad r(c). \}$$

$$U_P = \{a, b, c\}$$

$$B_P = \{ p(a, a), p(a, b), p(a, c), p(b, a), p(b, b), p(b, c), p(c, a), p(c, b), p(c, c), \\ q(a), q(b), q(c), \\ r(a), r(b), r(c) \}$$

$$I_P = \text{all subsets of } B_P$$

$$H_P = \{q(a), q(b), r(b), r(c), p(a, a), p(b, a), p(a, b), p(b, b), p(a, c), p(b, c)\}$$

$$T_P \uparrow 0 = \{q(a), q(b), r(b), r(c)\}$$

$$T_P \uparrow 1 = \{q(a), q(b), r(b), r(c)\} \cup \{p(a, a), p(b, a), p(a, b), p(b, b), p(a, c), p(b, c)\}$$

$$T_P \uparrow 2 = T_P \uparrow 1 = \text{lfp}(T_P) = H_P$$

11

A Fixpoint Semantics for Logic Programs: Example 2 (infinite)

$$P = \{ p(f(X)) \leftarrow p(X). \\ p(a). \\ q(a). \\ q(b). \}$$

$$U_P = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \dots\}$$

$$B_P = \{p(a), p(b), q(a), q(b), p(f(a)), p(f(b)), q(f(a)), \dots\}$$

$$I_P = \text{all subsets of } B_P$$

$$H_P = \{q(a), q(b), p(a)\} \cup \{p(f^n(a)) \mid n \in \mathcal{N}\}$$

where we define $f^n(a)$ to be f nested n times and then applied to a .

(i.e., $q(a), q(b), p(a), p(f(a)), p(f(f(a))), p(f(f(f(a))))$, ...)

$$T_P \uparrow 0 = \{p(a), q(a), q(b)\}$$

$$T_P \uparrow 1 = \{p(a), q(a), q(b), p(f(a))\}$$

$$T_P \uparrow 2 = \{p(a), q(a), q(b), p(f(a)), p(f(f(a)))\}$$

...

$$T_P \uparrow \omega = H_P$$

12

A Fixpoint Semantics for Logic Programs: Example 3 (infinite)

- Example:

$$P = \{ \text{nat}(0). \\ \text{nat}(s(X)) \leftarrow \text{nat}(X). \}$$

$$\text{sum}(0, X, X). \\ \text{sum}(s(X), Y, s(Z)) \leftarrow \text{sum}(X, Y, Z). \}$$

$$U_P = \{0\} \cup \{s(x) \mid x \in U_P\}$$

(i.e., $\{0, s(0), s(s(0)), s(s(s(0))), \dots\}$).

$$B_P = \{ \text{nat}(x) \mid x \in U_P \} \cup \{ \text{sum}(x, y, z) \mid x, y, z \in U_P \}$$

(i.e., $\{ \text{nat}(0), \text{nat}(s(0)), \text{nat}(s(s(0))), \dots \} \cup$
 $\{ \text{sum}(0, 0, 0), \text{sum}(s(0), 0, 0), \text{sum}(0, s(0), 0), \text{sum}(0, 0, s(0)), \dots \}$).

A Fixpoint Semantics for Logic Programs: Example 3 (infinite, cont.)

Constructing the least fixpoint of the T_P operator:

$$T_P \uparrow 0 = \{ \text{nat}(0) \} \cup \{ \text{sum}(0, x, x) \mid x \in U_P \}$$

$$T_P \uparrow 1 = T_P \uparrow 0 \cup \{ \text{nat}(s(0)) \} \\ \cup \{ \text{sum}(s(0), y, s(y)) \mid y \in U_P \}$$

$$T_P \uparrow 2 = T_P \uparrow 1 \cup \{ \text{nat}(s(s(0))) \} \\ \cup \{ \text{sum}(s(s(0)), y, s(s(y))) \mid y \in U_P \}$$

$$T_P \uparrow 3 = T_P \uparrow 2 \cup \{ \text{nat}(s(s(s(0)))) \} \\ \cup \{ \text{sum}(s(s(s(0))), y, s(s(s(y)))) \mid y \in U_P \}$$

...

$$T_P \uparrow \omega = \{ \text{nat}(x) \mid x \in U_P \} \cup \\ \{ \text{sum}(s^n(0), y, s^n(y)) \mid y \in U_P \wedge n \in \mathcal{N} \}$$

where we define $s^x(y)$ to be s nested x times and then applied to y .

Semantics – Equivalences

- (Characterization Theorem) [Van Emden and Kowalski]

A program P has a Herbrand model H_P such that :

- ◇ H_P is the least Herbrand Model of P .
- ◇ H_P is the least fixpoint of T_P ($\text{lfp } T_P$).
- ◇ $H_P = T_P \uparrow \omega$.

I.e., *least model semantics* (H_P) \equiv *fixpoint semantics* ($\text{lfp } T_P$)

- In addition, there is also an equivalence with the *operational semantics* (SLD-resolution):

- ◇ SLD-resolution answers “yes” to $a \in B_P \iff a \in H_P$.

- Because it gives us a way to directly build H_P (for finite models), the least fixpoint semantics can in some cases also be an operational semantics (e.g., for *datalog* in *deductive databases*).